Grassmannians & universal bundles

Recall: Grassmannian $G_n(\mathbb{R}^{n+k}) :=$ set of all n-dim'l vector subspaces of $\mathbb{R}^{n+k}$

Stiefel mfld $V_n(\mathbb{R}^{n+k}) :=$ set of all orthonormal n-frames in $\mathbb{R}^{n+k}$

(i.e. $v_1 \dots v_n \in \mathbb{R}^{n+k}$ s.t. $v_i \cdot v_j = \delta_{ij}$)

(could also consider all n-tuples of linearly indept vectors; retracts onto $V_n(\mathbb{R}^{n+k})$ by Gram-Schmidt).

with a fiber bundle $O(n) \to V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$

$(v_1, \dots, v_n) \longmapsto \mathrm{span}(v_1 \dots v_n)$

- Note: $V \mapsto V^\perp$ gives $G_n(\mathbb{R}^{n+k}) \simeq G_k(\mathbb{R}^{n+k})$

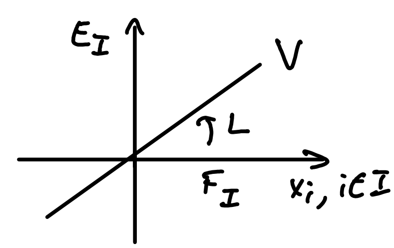
- $G_n(\mathbb{R}^{n+k})$ is a compact smooth manifold of dimension $n \cdot k$

Coordinate charts: for I a n-element subset of $\{1 \dots n+k\}$, $E_I = \{x_i = 0 \ \forall i \in I\}$ k-dim'l subspace

$U_I = \{\text{n-planes s.t. } V \cap E_I = 0\}$ $\quad F_I = E_I^\perp = \mathrm{span}(e_i, i \in I)$

then can view V as graph of a linear map $F_I \xrightarrow{L} E_I$

$\Rightarrow U_I \simeq \mathrm{Hom}(F_I, E_I) \simeq \mathbb{R}^{n \cdot k}$



eg. for $I = \{1 \dots n\}$, view $V \in U_I \subset G_n(\mathbb{R}^{n+k})$ as span of rows of

$$\begin{array}{cc} & \begin{array}{cc} n & k \end{array} \\ n & \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \quad L \quad \end{array} \right) \end{array}$$

- In fact, this shows: **Prop:** $T_{V_0} G_n(\mathbb{R}^{n+k}) \simeq \mathrm{Hom}(V_0, V_0^\perp) \quad \forall V_0 \in G_n$

(viewing V's st. $V \cap V_0^\perp = 0$ as graphs of linear maps $V_0 \to V_0^\perp$).

- Tautological rank n vector bundle $\tau \to G_n(\mathbb{R}^{n+k})$:

$$\begin{array}{c} \tau = \{(V, x) \in G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \,/\, x \in V\} \\ \downarrow \\ G_n(\mathbb{R}^{n+k}) \end{array}$$

(subbundle of trivial rank $(n+k)$ bundle).

(case $n=1 \Leftrightarrow \tau \to \mathbb{RP}^k$ seen previously).

- The relevance of this bundle comes from the Gauss map:

$M^n \subset \mathbb{R}^{n+k}$ smooth submfld $\Rightarrow \forall p \in M, \ T_pM \in G_n(\mathbb{R}^{n+k})$

The Gauss map is $g: M \longrightarrow G_n(\mathbb{R}^{n+k})$, $p \longmapsto T_pM$ (smooth map b/w manifolds)

Moreover, this is covered by a bundle map
$$\begin{array}{ccc} \tilde{g}: TM & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & G_n(\mathbb{R}^{n+k}) \end{array}$$
namely $\underset{\substack{\cap \\ M}}{\tilde{g}(p}, \underset{\substack{\cap \\ T_pM}}{x)} = (T_pM, x) \in \mathcal{T}$.

ie. $TM \simeq g^*\mathcal{T}$.

- In fact, all we need is: TM is a subbundle of a trivial bundle ($T\mathbb{R}^{n+k}{}_{|M}$).

The same property holds in much greater generality, and we have

<u>Prop:</u> $E \xrightarrow{\pi} B$ rank n vector bundle over a compact base B
$\Rightarrow$ for large enough k, $\exists$ map $B \xrightarrow{g} G_n(\mathbb{R}^{n+k})$ st. $E \simeq g^*\mathcal{T}$.

<u>Pf:</u>
- it suffices to show $E \simeq$ subbundle of a trivial bundle of rank $n+k$, ie find a continuous (resp. smooth) map $\hat{g}: E \to \mathbb{R}^{n+k}$ st. on each fiber $\hat{g}_b = g_{|E_b}: E_b \to \mathbb{R}^{n+k}$ is an injective linear map.

Then we just define
$$g: B \longrightarrow G_n(\mathbb{R}^{n+k})$$
$$b \longmapsto \hat{g}(E_b) \text{ (a } n\text{-dim}^l \text{ subspace of } \mathbb{R}^{n+k} \checkmark)$$
and by construction $E \simeq g^*\mathcal{T}$ (bundle map is
$$\begin{array}{ccc} E & \xrightarrow{(g,\hat{g})} & \mathcal{T} \\ \downarrow & & \downarrow \\ B & \xrightarrow[g]{} & G_n(\mathbb{R}^{n+k}) \end{array}$$
)

- Choose open covers $B = \bigcup_{i=1}^{r} V_i = \bigcup_{i=1}^{r} U_i$, $\overline{V_i} \subset U_i$, st. $E_{|U_i}$ trivial.

ie. $\exists h_i: E_{|U_i} \to \mathbb{R}^n$ st. h_i linear isom. on each fiber E_b, $b \in U_i$

also let $\chi_i: B \to \mathbb{R}$ st. $\chi_i = 1$ on V_i, 0 outside U_i

and let $h_i': E \to \mathbb{R}^n$: $\begin{cases} h_i'(x) = \lambda_i(\pi(x))\, h_i(x) & \text{if } x \in \pi^{-1}(U_i) \\ 0 & \text{else} \end{cases}$

fiberwise linear, isom. if $x \in \pi^{-1}(V_i)$.

Then $E \longrightarrow \mathbb{R}^{n \cdot r}$, $x \longmapsto (h_1'(x), \dots, h_r'(x))$ is the desired map

$\blacktriangle$

$\star$ If B noncompact, we need to consider instead the infinite grassmannian

$Gr_n(\mathbb{R}^\infty)$ = limit of $Gr_n(\mathbb{R}^n) \subset Gr_n(\mathbb{R}^{n+1}) \subset Gr_n(\mathbb{R}^{n+2}) \subset \dots$

($\mathbb{R}^\infty$ = limit of $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \dots$)

This carries tautological bundle $\tau \to Gr_n(\mathbb{R}^\infty)$
(= union of tautological bundles on $Gr_n(\mathbb{R}^{n+k})$'s).

We'll need to assume B <u>paracompact</u>, i.e. $\forall$ open cover (U_α) of B $\exists$ refinement (V_β) open cover that is locally finite

refinement: $\forall\, V_\beta \subset$ some U_α

locally finite: $\forall x\ \exists$ nbd $\cap$ finitely many V_β's.

eg: metric spaces, spaces which are $\cup$ countably many compact subsets

$E \to B$ (paracompact) fiber bundle $\Rightarrow$ $\exists$ loc. finite covering of B by countably many open subsets U_i st. $E_{|U_i}$ trivial.

Then: <u>Thm:</u> $E \xrightarrow{\pi} B$ rank n vector bundle over a paracompact base B
$\Rightarrow$ $\exists$ map $B \xrightarrow{g} G_n(\mathbb{R}^\infty)$ st. $E \simeq g^*\tau$.

<u>Pf:</u> as in compact case, except now get a map $\hat{g}: E \to \mathbb{R}^\infty$ (fiberwise injective linear) out of the maps $h_i': E \to \mathbb{R}^n$ (linear isoms inside V_i, zero outside U_i)
This does take values in $\mathbb{R}^\infty$ since local finiteness $\Rightarrow$ only finitely many nonzero coords. at a time; and then
$$g: B \longrightarrow G_n(\mathbb{R}^\infty)$$
$$b \longmapsto \hat{g}(E_b)$$
$\blacktriangle$

Moreover, this is canonical up to homotopy!! (even though construction seems ad hoc)

<u>Thm:</u> Any two bundle maps
$$\begin{array}{ccc} \tilde{f}, \tilde{g}: E & \longrightarrow & \tau \\ \downarrow & & \downarrow \\ f, g: B & \longrightarrow & G_n(\mathbb{R}^\infty) \end{array}$$
are homotopic through bundle maps.

<u>Pf:</u>
- first note such bundle maps $\tilde{f} \xleftrightarrow{1:1}$ maps $E \xrightarrow{\hat{f}} \mathbb{R}^\infty$ which are linear & injective on fibers
  (the information of the map $B \to G_n(\mathbb{R}^\infty)$ is given by $b \mapsto \hat{f}(E_b)$),
  so enough to show those are homotopic.
- if $\forall e \in E$, $e \notin$ zero section, $\hat{f}(e)$ and $\hat{g}(e)$ never negatively proportional:
  then just define $\hat{h}_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e)$. (continuous, fiberwise linear injective).
- reduce to this special case by trick:
  $\hat{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \dots) \overset{\text{(special case)}}{\sim} (\hat{f}_1, 0, \hat{f}_2, 0, \hat{f}_3, \dots) \overset{\text{(sp. case)}}{\sim} (0, \hat{g}_1, 0, \hat{g}_2, \dots) \overset{\text{(sp. case)}}{\sim} (\hat{g}_1, \hat{g}_2, \dots)$ $\blacktriangle$

<u>Corollary:</u> $\{\text{Rank } n \text{ real vector bundles over } B\}/\text{isomorphism}$
$\xleftrightarrow[1-1]{} \{\text{maps } B \to G_n(\mathbb{R}^\infty)\}/\text{homotopy}$.

$G_n(\mathbb{R}^\infty)$ is called the <u>classifying space</u> for real rank n vect bundles;
($= BO(n)$; univ. frame bundle $O(n) \to \underset{\text{contractible}}{V_n(\mathbb{R}^\infty)} \to G_n(\mathbb{R}^\infty) = BO(n)$)
the map $B \xrightarrow[f_E]{} G_n(\mathbb{R}^\infty)$ corresp. to a given bundle $E \to B$ is called <u>classifying map</u>.

* Now, given any class $c \in H^i(G_n(\mathbb{R}^\infty), \Lambda)$ $\rightsquigarrow$ $f_E^* c =: c(E) \in H^i(B, \Lambda)$
  (Λ: coeff. group or ring)

These are called <u>characteristic classes</u> of $E \to B$.
They are automatically <u>natural wrt. pullback</u>: $g : A \to B$, $E' \to E$ over g, $E' \simeq g^* E$
$\Rightarrow c(E') = g^* c(E)$.

Hence: we'll have to study $H^*(G_n(\mathbb{R}^\infty))$!

• <u>CW-structure on the Grassmannian:</u> generalizes that of $\mathbb{RP}^m = \underbrace{pt \cup (1\text{-cell}) \cup \ldots}_{\mathbb{RP}^{m-1}} \cup (m\text{-cell})$.

Cell structure of $\mathbb{RP}^m$ = write $\mathbb{R}^{m+1} \supset \mathbb{R}^m \supset \mathbb{R}^{m-1} \supset \cdots \supset \mathbb{R}^1$
& see which of these contains given line $\ell \subset \mathbb{R}^{m+1}$!
(attach hemisphere of S^m along double cover $S^{m-1} \to \mathbb{RP}^{m-1}$)

Similarly: $V \subset \mathbb{R}^m$ n-plane $\Rightarrow$
$0 \le \dim(V \cap \mathbb{R}^1) \le \dim(V \cap \mathbb{R}^2) \le \ldots \le \dim(V \cap \mathbb{R}^m) = n$
sequence of integers, increasing by at most 1 at a time: ie: n "jumps".

<u>Schubert symbol</u> $\sigma = (\sigma_1, \ldots, \sigma_n)$: integers st. $1 \le \sigma_1 < \sigma_2 < \ldots < \sigma_n \le m$.
$\rightsquigarrow e(\sigma) := \{V \in G_n(\mathbb{R}^m) \,/\, \dim(V \cap \mathbb{R}^{\sigma_i}) = i,\ \dim(V \cap \mathbb{R}^{\sigma_i - 1}) = i-1\}$

<u>Thm:</u> $e(\sigma)$ is an open cell of dim. $d(\sigma) = \sum_{i=1}^n (\sigma_i - i)$, and this decomposition gives $G_n(\mathbb{R}^m)$ the structure of a CW-complex.

<u>Pf:</u> • If $V \in e(\sigma)$ then can build a basis of V as follows:
→ let $v_1 \in \mathbb{R}^{\sigma_1}$ generate $V \cap \mathbb{R}^{\sigma_1}$ (note: $v_1 \notin \mathbb{R}^{\sigma_1 - 1}$ so $v_{1,\sigma_1} \neq 0$, can take $= 1$)
→ let $v_2 \in \mathbb{R}^{\sigma_2}$ st. v_1, v_2 basis of $V \cap \mathbb{R}^{\sigma_2}$ (can take $v_{2,\sigma_2} = 1$)
etc.

Hence $V \in e(\sigma) \Leftrightarrow V =$ column space of an $n\times m$ matrix of the form

($\Leftarrow$ obvious)

$$\begin{array}{c} \sigma_1\ \sigma_2 \cdots \sigma_n \\ \downarrow\ \ \downarrow \quad\ \downarrow \\ \begin{pmatrix} * \cdots * 1\, 0\, 0 \cdots\cdots 0 \\ * \cdots\cdots * 1\, 0 \cdots 0 \\ \vdots \\ * \cdots\cdots\cdots * 1\, 0 \ldots 0 \end{pmatrix} \end{array}$$

- To make things more canonical, apply Gram-Schmidt to get an orthonormal basis. Each vector stays in $\mathbb{R}^{\sigma_i}$, and last component > 0.
  Let $H^k :=$ half-space in $\mathbb{R}^k$: vectors with k^{th} component > 0.

<u>Lemma:</u> $V \in e(\sigma) \Rightarrow V$ has a unique orthonormal basis $(v_1, \ldots v_n)$ st. $v_i \in H^{\sigma_i}$

(v_1 unique; $v_2 \in \mathbb{R}^{\sigma_2} \cap (v_1^{\perp})$ 1-dim!, unique unit vector with last compt > 0; ...)

so: $e(\sigma) \underset{\text{homeo}}{\simeq} \underset{\substack{\text{Stiefel-mfld} \\ \text{(n-frames)}}}{V_n(\mathbb{R}^m)} \cap (H^{\sigma_1} \times \ldots \times H^{\sigma_n})$.

- let $E(\sigma) := V_n(\mathbb{R}^m) \cap (\overline{H^{\sigma_1}} \times \ldots \times \overline{H^{\sigma_n}})$ (i.e. allow last coord. to be zero!)
  closure of $e(\sigma)$ in $V_n(\mathbb{R}^m)$

<u>Lemma:</u> $E(\sigma) \simeq$ closed cell of dim. $d(\sigma) = \sum (\sigma_i - i)$.
The map $q: V_n(\mathbb{R}^m) \longrightarrow G_n(\mathbb{R}^m)$, $(v_1, \ldots, v_n) \longmapsto \mathrm{span}(v_1, \ldots, v_n)$ maps the interior of $E(\sigma)$ homeomorphically onto $e(\sigma)$.

<u>Idea:</u> We've already seen the last part. For <u>first</u> part, induction on n.
- choices of $v_1 = (\text{unit sphere}) \cap \overline{H}^{\sigma_1} =$ closed hemisphere of dim. $\sigma_1 - 1$
  so $E(\sigma_1) \simeq$ closed $(\sigma_1 - 1)$-cell.
- assume true for $E(\sigma_1, \ldots, \sigma_{n-1})$. Then
  for given $(v_1 \ldots v_{n-1}) \in E(\sigma_1 .. \sigma_{n-1})$, choices of v_n st. $(v_1 \ldots v_n) \in E(\sigma_1 .. \sigma_n)$
  $= (\text{unit sphere}) \cap \underbrace{(\mathrm{span}(v_1 .. v_{n-1})^{\perp}) \cap \overline{H}^{\sigma_n}}_{\text{closed } (\sigma_n - n+1)\text{-dim. halfspace}} =$ closed hemisphere of dim. $\sigma_n - n$.

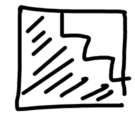
  gives fiber bundle $D^{\sigma_n - n} \to E(\sigma_1 .. \sigma_n) \to E(\sigma_1 .. \sigma_{n-1})$, $(v_1 .. v_n) \longmapsto (v_1 .. v_{n-1})$

  globally trivial (can work out triv. by seq of rotations)

- moreover, boundary of $E(\sigma)$ maps to $\bigcup e(\sigma')$ where $\sigma_i' \le \sigma_i \ \forall i = 1..n$
  (since vector $v_i \in \mathbb{R}^{\sigma_i}$ but might $\in \mathbb{R}^{\sigma_i'}$ for some $\sigma_i' < \sigma_i$).

<u>Corollary</u>: $G_n(\mathbb{R}^\infty)$ has structure of infinite CW-complex, with cells $e(\sigma)$, $1 \le \sigma_1 < \sigma_2 < \dots < \sigma_n$ (no upper bound on σ_n)

<u>Rmk</u>: lots of interesting combinatorics: eg.
$\#$cells in $G_n(\mathbb{R}^m)$ is $\binom{m}{n}$ (choose $\{\sigma_1, .., \sigma_n\} \subset \{1, .., m\}$)

$\#$ r-dim^l cells $=$ $\#$ partitions of r into at most n integers each $\le m-n$.
ie. diagrams  of area r.

$(0 \le \sigma_1 - 1 \le \sigma_2 - 2 \le \dots \le \sigma_n - n \le m-n, \quad \text{sum} = r)$.

for $G_n(\mathbb{R}^\infty)$, $\#$ r-cells $=$ partitions of r into $\le n$ integers (unbounded)

Next, understand $H^*(G_n(\mathbb{R}^\infty), \mathbb{Z}/2)$ ... useful observation:

<u>Rmk</u>: The attaching map $\partial E(\sigma) \to \bigcup_{\sigma' < \sigma} e(\sigma')$ has even degree onto all cells of dim. $d(\sigma) - 1$.

Indeed, the portion of $\partial E(\sigma_1 .. \sigma_n)$ mapping to $e(\sigma') = e(\sigma_1, \dots, \sigma_i - 1, \dots, \sigma_n)$ is $V_n(\mathbb{R}^m) \cap (H^{\sigma_1} \times \dots \times \mathbb{R}_*^{\sigma_i - 1} \times \dots \times H^{\sigma_n})$ ie. where vector $v_i \in \mathbb{R}^{\sigma_i - 1}$, last component $\ne 0$

Depending on whether $v_{i, \sigma_i - 1} > 0 \rightsquigarrow$ get $\mathrm{int}(E(\sigma'))$, maps homeo to $e(\sigma')$
$v_{i, \sigma_i - 1} < 0 \to$ same, under $v_i \leftrightarrow -v_i$.

(same as: r-cell of $\mathbb{RP}^{m-1}$ attaches onto $(r-1)$-cell by $S^{r-1} \xrightarrow[2:1]{} \mathbb{RP}^{r-1}$).

<u>Thm</u>: $H^*(G_n(\mathbb{R}^\infty), \mathbb{Z}/2)$ is a polynomial algebra $/\mathbb{Z}_2$ freely generated by the Stiefel-Whitney classes $w_1(\tau), \dots, w_n(\tau)$.

Start with <u>Lemma</u>: There are no polynomial relations among the $w_i(\tau)$.

<u>Pf</u>: if there were one, then the same relation would hold among $w_i(E)$ for <u>all</u> rank n bundles $E \to B$ (by naturality, since $\exists g$ st. $E \simeq g^*\tau$).
So: enough to find some rank n bundle whose S-W classes have no relations.
Consider $E = \tau \times \tau \times \dots \times \tau \simeq \pi_1^*\tau \oplus \pi_2^*\tau \oplus \dots \oplus \pi_n^*\tau$
$\downarrow$
$\mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty \qquad \pi_1, \dots, \pi_n : (\mathbb{P}^\infty)^n \to \mathbb{P}^\infty$.

Then $H^*((\mathbb{P}^\infty)^n, \mathbb{Z}/2) = \mathbb{Z}/2[a_1,..,a_n]$ by Künneth, and

$$w(E) = \prod_{k=1}^n (1 + w_1(\pi_k^* \tau)) = \prod_{k=1}^n (1 + \pi_k^*(w_1(\tau))) = \prod_{k=1}^n (1+a_k)$$

so $w_1(E) = a_1 + \dots + a_n$

$w_2(E) = a_1 a_2 + a_1 a_3 + \dots + a_{n-1} a_n$

...

$w_n(E) = a_1 \dots a_n$ elementary symm. polynomials

These are known to generate a free subalgebra of $\mathbb{Z}_2[a_1,..,a_n]$. ∎

Pf thm: counting argument: we now have $H^r(G_n(\mathbb{R}^\infty), \mathbb{Z}/2) \ni w_1(\tau)^{r_1} \dots w_n(\tau)^{r_n}$ all linearly indept.

$\forall (r_1 \dots r_n)$ s.t. $r_1 + 2r_2 + \dots + n r_n = r$.

Number of such $\Leftrightarrow$ partitions of r into at most n integers:

$0 \le r_n \le r_n + r_{n-1} \le \dots \le r_n + \dots + r_1$, sum $= r$.

We've seen before: this equals # r-cells of $G_n(\mathbb{R}^\infty)$ hence $=$ rank C^r hence $\ge$ rank H^r. Hence this is all of $H^r(G_n(\mathbb{R}^\infty), \mathbb{Z}/2)$, ie.

$\mathbb{Z}/2[w_1(\tau), \dots w_n(\tau)] \hookrightarrow H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}/2)$ by lemma

$\simeq$ by dim. counting

(Nb: we've seen $\delta = 0$, confirming rank (C^r) = rank (H^r) = # partitions. ∎

Rmk: • this pf was still conditional on existence of Stiefel-Whitney classes!!

• classifying map $g: \mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty \to G_n(\mathbb{R}^\infty)$ of bundle $\tau \times \dots \times \tau$ induces $g^*: H^*(G_n) \to H^*(\mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty) = \mathbb{Z}/2[a_1 \dots a_n]$

injective, mapping onto <u>symmetric</u> polynomials.

Corollary: Uniqueness of Stiefel Whitney classes satisfying axioms (if they exist).

Pf: axioms $\Rightarrow w_1(\tau_{\mathbb{RP}^1}) = a \Rightarrow w_1(\tau_{\mathbb{RP}^\infty}) = a$ (naturality for $\mathbb{RP}^1 \hookrightarrow \mathbb{P}^\infty$)

$\Rightarrow w(\tau \times \dots \times \tau) = \prod_1^n (1 + a_i)$ ($\oplus$, naturality)

$\Rightarrow g^* w(\tau_{G_n(\mathbb{R}^\infty)}) = \prod_1^n (1+a_i)$

since g^* injective, this determines uniquely $w(\tau)$ on $G_n(\mathbb{R}^\infty)$, but then also $w(E)$ $\forall E \to B$ using classifying map. ∎